

Energy and regularity dependent stability estimates for the Gel'fand inverse problem in multidimensions

M.I. Isaev and R.G. Novikov

Abstract

We prove new global Hölder-logarithmic stability estimates for the Gel'fand inverse problem at fixed energy in dimension $d \geq 3$. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. Comparisons with preceeding results in this direction are given.

1 Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \quad (1.1)$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d, \quad d \geq 2, \quad (1.2)$$

$$\text{with } \partial D \in C^2,$$

$$v \in \mathbb{L}^\infty(D). \quad (1.3)$$

Consider the map $\hat{\Phi} = \hat{\Phi}(E)$ such that

$$\hat{\Phi}(E)(\psi|_{\partial D}) = \frac{\partial\psi}{\partial\nu}|_{\partial D} \quad (1.4)$$

for all sufficiently regular solutions ψ of (1.1) in $\bar{D} = D \cup \partial D$, where ν is the outward normal to ∂D . Here we assume also that

$$E \text{ is not a Dirichlet eigenvalue for operator } -\Delta + v \text{ in } D. \quad (1.5)$$

The map $\hat{\Phi} = \hat{\Phi}(E)$ is called the Dirichlet-to-Neumann map and is considered as boundary measurements.

We consider the following inverse boundary value problem for equation (1.1):

Problem 1.1. Given $\hat{\Phi}$ for some fixed E , find v .

This problem can be considered as the Gelfand inverse boundary value problem for the Schrödinger equation at fixed energy (see [10], [23]). At zero energy this problem can be considered also as a generalization of the Calderon problem of the electrical impedance tomography (see [6], [23]). Problem 1.1 can be also

considered as an example of ill-posed problem: see [18], [4] for an introduction to this theory.

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness results and global reconstruction methods for Problem 1.1 were given for the first time in [23] in dimension $d \geq 3$ and in [5] in dimension $d = 2$.

Global logarithmic stability estimates for Problem 1.1 were given for the first time in [1] in dimension $d \geq 3$ and in [30] in dimension $d = 2$. A principal improvement of the result of [1] was given recently in [29] (for the zero energy case): stability of [29] optimally increases with increasing regularity of v .

For the Calderon problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [36] for $d \geq 3$ and in [21] for $d = 2$. Global logarithmic stability estimates for this problem were given for the first time in [1] for $d \geq 3$ and [19] for $d = 2$. Principal increasing of global stability of [1], [19] for the regular coefficient case was found in [29] for $d \geq 3$ and [34] for $d = 2$.

In addition, for the case of piecewise constant or piecewise real analytic conductivity the first uniqueness results for the Calderon problem in dimension $d \geq 2$ were given in [7], [16]. Lipschitz stability estimate for the case of piecewise constant conductivity was proved in [2] and additional studies in this direction were fulfilled in [33].

Due to [20] the logarithmic stability results of [1], [19] with their principal effectivization of [29], [34] are optimal (up to the value of the exponent). An extension of the instability estimates of [20] to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was given in [12].

On the other hand, it was found in [25], [26] (see also [28], [31]) that for inverse problems for the Schrödinger equation at fixed energy E in dimension $d \geq 2$ (like Problem 1.1) there is a Hölder stability modulo an error term rapidly decaying as $E \rightarrow +\infty$ (at least for the regular coefficient case). In addition, for Problem 1.1 for $d = 3$, global energy dependent stability estimates changing from logarithmic type to Hölder type for high energies were given in [15]. However, there is no efficient stability increasing with respect to increasing coefficient regularity in these results of [15]. An additional study, motivated by [15], [29], was given in [22].

In the present work we give new global Hölder-logarithmic stability estimates for Problem 1.1 in dimension $d \geq 3$ for the regular coefficient case, see Theorem 2.1 and Remark 2.6. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. In particular cases, our new estimates become coherent (although less strong) with respect to results of [29], [26], see Remarks 2.2, 2.3. In general, our new estimates give some synthesis of several important preceeding results.

2 Stability estimates

In this section we assume for simplicity that

$$v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d, \text{ supp } v \subset D, \quad (2.1)$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v : \partial^J v \in L^1(\mathbb{R}^d), |J| \leq m\}, \quad m \in \mathbb{N} \cup 0, \quad (2.2)$$

where

$$J \in (\mathbb{N} \cup 0)^d, \quad |J| = \sum_{i=1}^d J_i, \quad \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}. \quad (2.3)$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{L^1(\mathbb{R}^d)}. \quad (2.4)$$

Let

$$\begin{aligned} \|A\| \text{ denote the norm of an operator} \\ A : \mathbb{L}^\infty(\partial D) \rightarrow \mathbb{L}^\infty(\partial D). \end{aligned} \quad (2.5)$$

We recall that if v_1, v_2 are potentials satisfying (1.3), (1.5) for some fixed E , then

$$\hat{\Phi}_2(E) - \hat{\Phi}_1(E) \text{ is a compact operator in } \mathbb{L}^\infty(\partial D), \quad (2.6)$$

where $\hat{\Phi}_1, \hat{\Phi}_2$ are the DtN maps for v_1, v_2 , respectively, see [23], [27]. Note also that (2.1) \Rightarrow (1.3).

Let

$$s_0 = \frac{m-d}{m}, \quad s_1 = \frac{m-d}{d}, \quad s_2 = m-d. \quad (2.7)$$

Theorem 2.1. *Let D satisfy (1.2), where $d \geq 3$. Let v_1, v_2 satisfy (2.1) and (1.5) for some fixed real E . Let $\|v_j\|_{m,1} \leq N$, $j = 1, 2$, for some $N > 0$. Let $\hat{\Phi}_1(E)$ and $\hat{\Phi}_2(E)$ denote the DtN maps for v_1 and v_2 , respectively. Then*

$$\|v_2 - v_1\|_{L^\infty(D)} \leq C_1 (\ln(3 + \delta^{-1}))^{-s}, \quad 0 < s \leq s_1, \quad (2.8)$$

where $C_1 = C_1(N, D, m, s, E) > 0$, $\delta = \|\hat{\Phi}_2(E) - \hat{\Phi}_1(E)\|$ is defined according to (2.5). In addition, for $E \geq 0$, $\tau \in (0, 1)$ and any $s \in [0, s_1]$,

$$\|v_2 - v_1\|_{L^\infty(D)} \leq C_2(1 + \sqrt{E})\delta^\tau + C_3(1 + \sqrt{E})^{s-s_1} (\ln(3 + \delta^{-1}))^{-s}, \quad (2.9)$$

where $C_2 = C_2(N, D, m, \tau) > 0$ and $C_3 = C_3(N, D, m, \tau) > 0$.

Remark 2.1. Estimate (2.8) for $s = s_0$ is a variation of the result of [1] (see also [29], [13]). One can see that estimate (2.8), $s = s_1$, of Theorem 2.1 is more strong (as much as s_1 is greater than s_0) than the aforementioned result going back to [1].

Remark 2.2. Estimate (2.8) for $s = s_2$, $E = 0$, $d = 3$ was proved in [29]. One can see that this estimate of [29] is more strong (as much as s_2 is greater than s_1) than estimate (2.8), $s = s_1$, of Theorem 2.1 for $E = 0$, $d = 3$.

Remark 2.3. Using results of [26] one can obtain estimate (2.9) for $s = 0$, $d = 3$, with s_2 in place of s_1 , for sufficiently great E with respect to N . One can see that for this particular case the aforementioned corollary of [26] is more strong (as much as s_2 is greater than s_1) than estimate (2.9) of Theorem 2.1.

Remark 2.4. In a similar way with results of [13], [14], estimates (2.8), (2.9) can be extended to the case when we do not assume that condition (1.5) is fulfilled and consider an appropriate impedance boundary map instead of the Dirichlet-to-Neumann map.

Remark 2.5. Concerning two-dimensional analogs of results of Theorem 2.1, see [25], [31], [34], [35].

Remark 2.6. Actually, in the proof of Theorem 2.1 we obtain the following estimate (see formula (4.19)):

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \leq C_4 \sqrt{E + \rho^2} e^{2\rho L} \delta + C_5 (E + \rho^2)^{-s_1/2}, \quad (2.10)$$

where $L = \max_{x \in \partial D} |x|$, $C_4 = C_4(N, D, m) > 0$, $C_5 = C_5(N, D, m) > 0$ and parameter $\rho > 0$ is such that $E + \rho^2$ is sufficiently large: $E + \rho^2 \geq C_6(N, D, m)$. Estimates of Theorem 2.1 follow from estimate (2.10).

The proof of Theorem 2.1 and estimate (2.10) is given in Section 4 and is based on results recalled in Section 3. Actually, this proof is technically very similar to the proof of estimate (2.8) for $s = s_0$, see [1], [29], [13]. Possibility of such a proof of estimate (2.8) for $s = s_1$, $E = 0$ was mentioned, in particular, in [32].

3 Faddeev functions

We consider the Faddeev functions G , ψ , h (see [8], [9], [11], [23]):

$$G(x, k) = e^{ikx} g(x, k), \quad g(x, k) = -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \quad (3.1)$$

$$\psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k) v(y) \psi(y, k) dy, \quad (3.2)$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d$, $\text{Im } k \neq 0$, $d \geq 3$,

$$h(k, l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx} v(x) \psi(x, k) dx, \quad (3.3)$$

where

$$k, l \in \mathbb{C}^d, \quad k^2 = l^2, \quad \text{Im } k = \text{Im } l \neq 0. \quad (3.4)$$

One can consider (3.2), (3.3) assuming that

$$\begin{aligned} v &\text{ is a sufficiently regular function on } \mathbb{R}^d \\ &\text{ with sufficient decay at infinity.} \end{aligned} \quad (3.5)$$

For example, in connection with Problem 1.1, one can consider (3.2), (3.3) assuming that

$$v \in \mathbb{L}^\infty(D), \quad v \equiv 0 \text{ on } \mathbb{R} \setminus D. \quad (3.6)$$

We recall that (see [8], [9], [11], [23]):

- The function G satisfies the equation

$$(\Delta + k^2)G(x, k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d; \quad (3.7)$$

- Formula (3.2) at fixed k is considered as an equation for

$$\psi = e^{ikx} \mu(x, k), \quad (3.8)$$

where μ is sought in $\mathbb{L}^\infty(\mathbb{R}^d)$;

- As a corollary of (3.2), (3.1), (3.7), ψ satisfies (1.1) for $E = k^2$;
- The Faddeev functions G , ψ , h are (non-analytic) continuation to the complex domain of functions of the classical scattering theory for the Schrödinger equation (in particular, h is a generalized "scattering" amplitude).

In addition, G , ψ , h in their zero energy restriction, that is for $E = 0$, were considered for the first time in [3]. The Faddeev functions G , ψ , h were, actually, rediscovered in [3].

Let

$$\begin{aligned} \Sigma_E &= \{k \in \mathbb{C}^d : k^2 = k_1^2 + \dots + k_d^2 = E\}, \\ \Theta_E &= \{k \in \Sigma_E, l \in \Sigma_E : \operatorname{Im} k = \operatorname{Im} l\}, \\ |k| &= (|\operatorname{Re} k|^2 + |\operatorname{Im} k|^2)^{1/2}. \end{aligned} \quad (3.9)$$

Under the assumptions of Theorem 2.1, we have that:

$$\mu(x, k) \rightarrow 1 \quad \text{as} \quad |k| \rightarrow \infty \quad (3.10)$$

and, for any $\sigma > 1$,

$$|\mu(x, k)| \leq \sigma \quad \text{for} \quad |k| \geq r_1(N, D, m, \sigma), \quad (3.11)$$

where $x \in \mathbb{R}^d$, $k \in \Sigma_E$;

$$\hat{v}(p) = \lim_{\substack{(k, l) \in \Theta_E, k - l = p \\ |\operatorname{Im} k| = |\operatorname{Im} l| \rightarrow \infty}} h(k, l) \quad \text{for any } p \in \mathbb{R}^d, \quad (3.12)$$

$$\begin{aligned} |\hat{v}(p) - h(k, l)| &\leq \frac{c_1(D, m)N^2}{(E + \rho^2)^{1/2}} \quad \text{for } (k, l) \in \Theta_E, \quad p = k - l, \\ |\operatorname{Im} k| = |\operatorname{Im} l| &= \rho, \quad E + \rho^2 \geq r_2(N, D, m), \\ p^2 &\leq 4(E + \rho^2), \end{aligned} \quad (3.13)$$

where

$$\hat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \quad (3.14)$$

Results of the type (3.10), (3.11) go back to [3]. For more information concerning (3.11) see estimate (4.11) of [13]. Results of the type (3.12), (3.13) (with less precise right-hand side in (3.13)) go back to [11]. Estimate (3.13) follows, for example, from formulas (3.2), (3.3) and the estimate

$$\begin{aligned} \|\Lambda^{-s} g(k) \Lambda^{-s}\|_{\mathbb{L}^2(\mathbb{R}^d) \rightarrow \mathbb{L}^2(\mathbb{R}^d)} &= O(|k|^{-1}) \\ \text{as } |k| &\rightarrow \infty, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d, \end{aligned} \quad (3.15)$$

for $s > 1/2$, where $g(k)$ denotes the integral operator with the Schwartz kernel $g(x-y, k)$ and Λ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$.

Estimate (3.15) was formulated, first, in [17] for $d \geq 3$. Concerning proof of (3.15), see [37].

In addition, we have that:

$$\begin{aligned} h_2(k, l) - h_1(k, l) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \psi_1(x, -l)(v_2(x) - v_1(x))\psi_2(x, k)dx \\ &\quad \text{for } (k, l) \in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0, \\ &\quad \text{and } v_1, v_2 \text{ satisfying (3.5),} \end{aligned} \quad (3.16)$$

$$\begin{aligned} h_2(k, l) - h_1(k, l) &= (2\pi)^{-d} \int_{\partial D} \psi_1(x, -l) \left[(\hat{\Phi}_2 - \hat{\Phi}_1) \psi_2(\cdot, k) \right] (x) dx \\ &\quad \text{for } (k, l) \in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0, \\ &\quad \text{and } v_1, v_2 \text{ satisfying (1.5), (3.6),} \end{aligned} \quad (3.17)$$

and, under assumptions of Theorem 2.1,

$$\begin{aligned} |\hat{v}_1(p) - \hat{v}_2(p) - h_1(k, l) + h_2(k, l)| &\leq \frac{c_2(D, m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E + \rho^2)^{1/2}} \\ &\quad \text{for } (k, l) \in \Theta_E, \quad p = k - l, \quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \\ &\quad E + \rho^2 \geq r_3(N, D, m), \quad p^2 \leq 4(E + \rho^2), \end{aligned} \quad (3.18)$$

where h_j, ψ_j denote h and ψ of (3.3) and (3.2) for $v = v_j$, and $\hat{\Phi}_j$ denotes the Dirichlet-to-Neumann map for $v = v_j$, where $j = 1, 2$.

Formulas (3.16), (3.17) were given in [24], [27]. Estimate (3.18) follows from (3.2), (3.15), (3.16) in a similar way as estimate (3.13) follows from (3.2), (3.3), (3.15).

4 Proof of Theorem 2.1

Let

$$\begin{aligned} \mathbb{L}_\mu^\infty(\mathbb{R}^d) &= \{u \in \mathbb{L}^\infty(\mathbb{R}^d) : \|u\|_\mu < +\infty\}, \\ \|u\|_\mu &= \operatorname{ess\,sup}_{p \in \mathbb{R}^d} (1 + |p|)^\mu |u(p)|, \quad \mu > 0. \end{aligned} \quad (4.1)$$

Note that

$$\begin{aligned} w \in \mathbb{W}^{m,1}(\mathbb{R}^d) &\implies \hat{w} \in \mathbb{L}_\mu^\infty(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d), \\ \|\hat{w}\|_\mu &\leq c_3(m, d)\|w\|_{m,1} \quad \text{for } \mu = m, \end{aligned} \quad (4.2)$$

where $\mathbb{W}^{m,1}, \mathbb{L}_\mu^\infty$ are the spaces of (2.2), (4.1),

$$\hat{w}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} w(x) dx, \quad p \in \mathbb{R}^d. \quad (4.3)$$

Using the inverse Fourier transform formula

$$w(x) = \int_{\mathbb{R}^d} e^{-ipx} \hat{w}(p) dp, \quad x \in \mathbb{R}^d, \quad (4.4)$$

we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} &\leq \sup_{x \in \bar{D}} \left| \int_{\mathbb{R}^d} e^{-ipx} (\hat{v}_2(p) - \hat{v}_1(p)) dp \right| \leq \\ &\leq I_1(r) + I_2(r) \quad \text{for any } r > 0, \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} I_1(r) &= \int_{|p| \leq r} |\hat{v}_2(p) - \hat{v}_1(p)| dp, \\ I_2(r) &= \int_{|p| \geq r} |\hat{v}_2(p) - \hat{v}_1(p)| dp. \end{aligned} \quad (4.6)$$

Using (4.2), we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \leq 2c_3(m, d)N(1 + |p|)^{-m}, \quad p \in \mathbb{R}^d. \quad (4.7)$$

Due to (3.18), we have that

$$\begin{aligned} |\hat{v}_2(p) - \hat{v}_1(p)| &\leq |h_2(k, l) - h_1(k, l)| + \frac{c_2(D, m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E + \rho^2)^{1/2}}, \\ &\text{for } (k, l) \in \Theta_E, \quad p = k - l, \quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \\ &E + \rho^2 \geq r_3(N, D, m), \quad p^2 \leq 4(E + \rho^2). \end{aligned} \quad (4.8)$$

Let

$$\begin{aligned} c_4 &= (2\pi)^{-d} \int_{\partial D} dx, \quad L = \max_{x \in \partial D} |x|, \\ \delta &= \|\hat{\Phi}_2(E) - \hat{\Phi}_1(E)\|, \end{aligned} \quad (4.9)$$

where $\|\hat{\Phi}_2(E) - \hat{\Phi}_1(E)\|$ is defined according to (2.5).

Due to (3.17), we have that

$$\begin{aligned} |h_2(k, l) - h_1(k, l)| &\leq c_4 \|\psi_1(\cdot, -l)\|_{\mathbb{L}^\infty(\partial D)} \delta \|\psi_2(\cdot, k)\|_{\mathbb{L}^\infty(\partial D)}, \\ &(k, l) \in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0. \end{aligned} \quad (4.10)$$

Using (3.11), we find that

$$\begin{aligned} \|\psi(\cdot, k)\|_{\mathbb{L}^\infty(\partial D)} &\leq \sigma \exp\left(|\operatorname{Im} k|L\right), \\ k &\in \Sigma_E, \quad |k| \geq r_1(N, D, m, \sigma). \end{aligned} \quad (4.11)$$

Here and bellow in this section the constant σ is the same that in (3.11).

Combining (4.10) and (4.11), we obtain that

$$\begin{aligned} |h_2(k, l) - h_1(k, l)| &\leq c_4 \sigma^2 e^{2\rho L} \delta, \quad \text{for } (k, l) \in \Theta_E, \\ \rho &= |\operatorname{Im} k| = |\operatorname{Im} l|, \\ E + \rho^2 &\geq r_1^2(N, D, m, \sigma). \end{aligned} \quad (4.12)$$

Using (4.8), (4.12), we get that

$$\begin{aligned} |\hat{v}_2(p) - \hat{v}_1(p)| &\leq c_4 \sigma^2 e^{2\rho L} \delta + \frac{c_2(D, m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E + \rho^2)^{1/2}}, \\ p &\in \mathbb{R}^d, \quad p^2 \leq 4(E + \rho^2), \quad E + \rho^2 \geq \max\{r_1^2, r_3\}. \end{aligned} \quad (4.13)$$

Let

$$\varepsilon = \left(\frac{1}{2c_2(D, m)Nc_5} \right)^{1/d}, \quad c_5 = \int_{p \in \mathbb{R}^d, |p| \leq 1} dp, \quad (4.14)$$

and $r_4(N, D, m, \sigma) > 0$ be such that

$$E + \rho^2 \geq r_4(N, D, m, \sigma) \implies \begin{cases} E + \rho^2 \geq r_1^2(N, D, m, \sigma), \\ E + \rho^2 \geq r_3(N, D, m), \\ \left(\varepsilon(E + \rho^2)^{\frac{1}{2d}} \right)^2 \leq 4(E + \rho^2). \end{cases} \quad (4.15)$$

Let

$$c_6 = \int_{p \in \mathbb{R}^d, |p|=1} dp. \quad (4.16)$$

Using (4.6), (4.13), we get that

$$\begin{aligned} I_1(r) &\leq c_5 r^d \left(c_4 \sigma^2 e^{2\rho L} \delta + \frac{c_2(D, m)N \|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E + \rho^2)^{1/2}} \right), \\ r &> 0, \quad r^2 \leq 4(E + \rho^2), \\ E + \rho^2 &\geq r_4(N, D, m, \sigma). \end{aligned} \quad (4.17)$$

Using (4.6), (4.7), we find that, for any $r > 0$,

$$I_2(r) \leq 2c_3(m, d)Nc_6 \int_r^{+\infty} \frac{dt}{t^{m-d+1}} \leq \frac{2c_3(m, D)Nc_6}{m-d} \frac{1}{r^{m-d}}. \quad (4.18)$$

Combining (4.5), (4.17), (4.18) for $r = \varepsilon(E + \rho^2)^{\frac{1}{2d}}$ and (4.15), we get that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} &\leq c_7(N, D, m, \sigma) \sqrt{E + \rho^2} e^{2\rho L} \delta + \\ &+ c_8(N, D, m) (E + \rho^2)^{-\frac{m-d}{2d}} + \frac{1}{2} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)}, \\ E + \rho^2 &\geq r_4(N, D, m, \sigma). \end{aligned} \quad (4.19)$$

Let $\tau' \in (0, 1)$ and

$$\beta = \frac{1 - \tau'}{2L}, \quad \rho = \beta \ln(3 + \delta^{-1}), \quad (4.20)$$

where δ is so small that $E + \rho^2 \geq r_4(N, D, m, \sigma)$. Then due to (4.19), we have that

$$\begin{aligned} \frac{1}{2} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} &\leq \\ &\leq c_7(N, D, m, \sigma) \left(E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{1/2} (3 + \delta^{-1})^{2\beta L} \delta + \\ &+ c_8(N, D, m) \left(E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{-\frac{m-d}{2d}} = \\ &= c_7(N, D, m, \sigma) \left(E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{1/2} (1 + 3\delta)^{1-\tau'} \delta^{\tau'} + \\ &+ c_8(N, D, m) \left(E + (\beta \ln(3 + \delta^{-1}))^2 \right)^{-\frac{m-d}{2d}}, \end{aligned} \quad (4.21)$$

where τ', β and δ are the same as in (4.20).

Using (4.21), we obtain that

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \leq c_9(N, D, E, m, \sigma, \tau') (\ln(3 + \delta^{-1}))^{-\frac{m-d}{d}} \quad (4.22)$$

for $\delta = \|\hat{\Phi}_2 - \hat{\Phi}_1\| \leq \delta_1(N, D, E, m, \sigma, \tau')$, where δ_1 is a sufficiently small positive constant. Estimate (4.22) in the general case (with modified c_9) follows from (4.22) for $\delta \leq \delta_1(N, D, E, m, \sigma, \tau')$ and the property that

$$\|v_j\|_{\mathbb{L}^\infty(D)} \leq c_{10}(D, m)N. \quad (4.23)$$

This completes the proof of (2.8).

If $E \geq 0$ then there is a constant $\delta_2 = \delta_2(N, D, m, \sigma, \tau') > 0$ such that

$$\delta \in (0, \delta_2) \implies \begin{cases} E + (\beta \ln(3 + \delta^{-1}))^2 \geq r_4(N, D, m, \sigma), \\ E + (\beta \ln(3 + \delta^{-1}))^2 \leq \left((1 + \sqrt{E})\beta \ln(3 + \delta^{-1})\right)^2, \\ \beta \ln(3 + \delta^{-1}) \geq 1, \end{cases} \quad (4.24)$$

where β is the same as in (4.20). Combining (4.21), (4.24), we obtain that for $s \in [0, (m-d)/d]$, $\tau \in (0, \tau')$ and $\delta \in (0, \delta_2)$ the following estimate holds:

$$\|v_2 - v_1\|_{L^\infty(D)} \leq c_{11}(1 + \sqrt{E})\delta^\tau + c_{12}(1 + \sqrt{E})^{s-\frac{m-d}{d}} (\ln(3 + \delta^{-1}))^{-s}, \quad (4.25)$$

where constants $c_{11}, c_{12} > 0$ depend only on N, D, m, σ, τ' and τ .

Estimate (4.25) in the general case (with modified c_{11} and c_{12}) follows from (4.25) for $\delta \leq \delta_2(N, D, m, \sigma, \tau')$ and (4.23).

This completes the proof of (2.9)

Acknowledgements

The second author was partially supported by the Russian Federation Government grant No. 2010-220-01-077.

References

- [1] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. 27, 1988, 153-172.
- [2] G. Alessandrini, S. Vassella, *Lipschitz stability for the inverse conductivity problem*, Adv. in Appl. Math. 35, 2005, no.2, 207-241.
- [3] R. Beals and R. Coifman, *Multidimensional inverse scattering and nonlinear partial differential equations*, Proc. Symp. Pure Math., 43, 1985, 45-70.
- [4] L. Beilina, M.V. Klibanov, *Approximate global convergence and adaptivity for coefficient inverse problems*, Springer (New York), 2012. 407 pp.
- [5] A. L. Bukhgeim, *Recovering a potential from Cauchy data in the two-dimensional case*, J. Inverse Ill-Posed Probl. 16, 2008, no. 1, 19-33.

- [6] Calderón, A.P., *On an inverse boundary problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matematica, Rio de Janeiro, 1980, 61-73.
- [7] V. Druskin, *The unique solution of the inverse problem in electrical surveying and electrical well logging for piecewise-constant conductivity*, Physics of the Solid Earth 18(1), 1982, 51-53.
- [8] L.D. Faddeev, *Growing solutions of the Schrödinger equation*, Dokl. Akad. Nauk SSSR, 165, N.3, 1965, 514-517 (in Russian); English Transl.: Sov. Phys. Dokl. 10, 1966, 1033-1035.
- [9] L.D. Faddeev, *The inverse problem in the quantum theory of scattering. II*, Current problems in mathematics, Vol. 3, 1974, pp. 93180, 259. Akad. Nauk SSSR Vsesojuz. Inst. Nauch. i Tehn. Informacii, Moscow(in Russian); English Transl.: J.Sov. Math. 5, 1976, 334-396.
- [10] I.M. Gelfand, *Some problems of functional analysis and algebra*, Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, pp.253-276.
- [11] G.M. Henkin and R.G. Novikov, *The $\bar{\partial}$ -equation in the multidimensional inverse scattering problem*, Uspekhi Mat. Nauk 42(3), 1987, 93-152 (in Russian); English Transl.: Russ. Math. Surv. 42(3), 1987, 109-180.
- [12] M.I. Isaev, *Exponential instability in the Gelfand inverse problem on the energy intervals*, J. Inverse Ill-Posed Probl., Vol. 19(3), 2011, 453-473; e-print arXiv: 1012.2193.
- [13] M.I. Isaev, R.G. Novikov *Stability estimates for determination of potential from the impedance boundary map*, e-print arXiv:1112.3728.
- [14] M.I. Isaev, R.G. Novikov *Reconstruction of a potential from the impedance boundary map*, e-print arXiv:1204.0076.
- [15] V. Isakov, *Increasing stability for the Schrödinger potential from the Dirichlet-to-Neumann map*, Discrete Contin. Dyn. Syst. Ser. S 4, 2011, no. 3, 631-640.
- [16] R. Kohn, M. Vogelius, *Determining conductivity by boundary measurements II*, Interior results, Comm. Pure Appl. Math. 38, 1985, 643-667.
- [17] R.B. Lavine and A.I. Nachman, *On the inverse scattering transform of the n -dimensional Schrödinger operator* Topics in Soliton Theory and Exactly Solvable Nonlinear Equations ed M Ablowitz, B Fuchssteiner and M Kruskal (Singapore: World Scientific), 1987, pp 33-44.
- [18] M.M. Lavrentev, V.G. Romanov, S.P. Shishatskii, *Ill-posed problems of mathematical physics and analysis*, Translated from the Russian by J. R. Schulenberger. Translation edited by Lev J. Leifman. Translations of Mathematical Monographs, 64. American Mathematical Society, Providence, RI, 1986. vi+290 pp.

- [19] L. Liu, *Stability Estimates for the Two-Dimensional Inverse Conductivity Problem*, Ph.D. thesis, Department of Mathematics, University of Rochester, New York, 1997.
- [20] N. Mandache, *Exponential instability in an inverse problem for the Schrödinger equation*, Inverse Problems. 17, 2001, 1435-1444.
- [21] A. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann. Math. 143, 1996, 71-96.
- [22] S. Nagayasu, G. Uhlmann, J.-N. Wang, *Increasing stability in an inverse problem for the acoustic equation*, e-print arXiv:1110.5145
- [23] R.G. Novikov, *Multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$* Funkt. Anal. Prilozhen. 22(4), 1988, 11-22 (in Russian); Engl. Transl. Funct. Anal. Appl. 22, 1988, 263-272.
- [24] R.G. Novikov, *$\bar{\partial}$ -method with nonzero background potential. Application to inverse scattering for the two-dimensional acoustic equation*, Comm. Partial Differential Equations 21, 1996, no. 3-4, 597-618.
- [25] R.G. Novikov, *Rapidly converging approximation in inverse quantum scattering in dimension 2*, Physics Letters A 238, 1998, 73-78.
- [26] R.G. Novikov, *The $\bar{\partial}$ -approach to approximate inverse scattering at fixed energy in three dimensions*. IMRP Int. Math. Res. Pap. 2005, no. 6, 287-349.
- [27] R.G. Novikov, *Formulae and equations for finding scattering data from the Dirichlet-to-Neumann map with nonzero background potential*, Inverse Problems 21, 2005, 257-270.
- [28] R.G. Novikov, *The $\bar{\partial}$ -approach to monochromatic inverse scattering in three dimensions*, J. Geom. Anal 18, 2008, 612-631.
- [29] R.G. Novikov, *New global stability estimates for the Gel'fand-Calderon inverse problem*, Inverse Problems 27, 2011, 015001(21pp); e-print arXiv:1002.0153.
- [30] R. Novikov and M. Santacesaria, *A global stability estimate for the Gelfand-Calderon inverse problem in two dimensions*, J.Inverse Ill-Posed Probl., Volume 18, Issue 7, 2010, Pages 765-785; e-print arXiv: 1008.4888.
- [31] R. Novikov and M. Santacesaria, *Monochromatic Reconstruction Algorithms for Two-dimensional Multi-channel Inverse Problems*, International Mathematics Research Notes, 2012, doi: 10.1093/imrn/rns025.
- [32] V.P. Palamodov, private communication of February 2011.
- [33] L. Rondi, *A remark on a paper by Alessandrini and Vessella*, Adv. in Appl. Math. 36 (1), 2006, 67-69.
- [34] M. Santacesaria, *New global stability estimates for the Calderon inverse problem in two dimensions*, e-print: hal-00628403.

- [35] M. Santacesaria, *Stability estimates for an inverse problem for the Schrödinger equation at negative energy in two dimensions*, e-print: hal-00688457.
- [36] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. 125, 1987, 153-169.
- [37] R. Weder, *Generalized limiting absorption method and multidimensional inverse scattering theory*, Mathematical Methods in the Applied Sciences, 14, 1991, 509-524.

M.I. Isaev

Centre de Mathématiques Appliquées, Ecole Polytechnique,
 91128 Palaiseau, France
 Moscow Institute of Physics and Technology,
 141700 Dolgoprudny, Russia
 e-mail: isaev.m.i@gmail.com

R.G. Novikov

Centre de Mathématiques Appliquées, Ecole Polytechnique,
 91128 Palaiseau, France
 Institute of Earthquake Prediction Theory and Math. Geophysics RAS,
 117997 Moscow, Russia
 e-mail: novikov@cmap.polytechnique.fr